Sequential change-point detection in time series: An adjusted-range-based self-normalization approach

Meiting Zhu†, Yongmiao Hong‡, Jiajing Sun§, Oliver Linton¶

School of Economics, Xiamen University†
Center for Forecasting Science, Chinese Academy of Sciences‡
School of Economics and Management, University of Chinese Academy of Sciences§
Faculty of Economics, University of Cambridge§

August 2025

Table of Contents

1. Introduction
2. Problem setup and sequential monitoring scheme
3. Asymptotic properties of the RSMS
4. Simulations
5. Empirical analysis
6. Conclusion
7. References

Introduction

- The ability to update models in real time to reflect the evolving scope of real-world data is a fundamental task in statistics. Current one-shot tests often lead to incorrect rejections.[1]
- **Question**: "Is yesterday's model still relevant for today's data?"(Chu, Stinchcombe, and White 1996)
- **Aim**: To develop a sequential change-point detection algorithm, which enables swift, real-time detection of model changes in various dynamic environments such as finance, environmental science, epidemiology, and other fields.

[1] Robbins (1970) points out that as the number of applications increases, the repeated utilization of such tests leads to a procedure that progressively rejects a true null hypothesis of no change with a probability approaching 1.

Introduction

- A sequential monitoring scheme typically begins with an initial period of length m as a training sample. The newly observed data points are then compared against this training sample.
- In the absence of structural breaks, the training and testing samples should exhibit similar probabilistic properties.
- A common approach involves computing a cumulative sum (CUSUM) of functionals derived from the testing sample. This CUSUM statistic is then standardized using the long-run variance (LRV), estimated from the training sample.[2]

[2] This method is widely adopted; see Chu, Stinchcombe, and White (1996), Gut and Steinebach (2002), Berkes et al. (2004), Zeileis et al. (2005), Gombay and Horváth (2009), Gombay and Serban (2009), among others.

Introduction

- Many LRV estimation methods require user-specified parameters, such as:
  - Block size in block bootstrap methods.
  - Kernel and bandwidth choices in kernel-based approaches.
- To mitigate parameter selection issues in LRV estimation, the self-normalization (SN) approach was introduced by Shao (2010) as an extension of the idea in Lobato 2001.
  - The self-normalized LRV estimator is an inconsistent estimator, but it is stochastically proportional to the true LRV and free of nuisance parameters.
  - It can be used to standardize test statistics without requiring explicit parameter selection.

Introduction

- However, Shao's (2010) SN method exhibits a "better size and less power" issue
    - The Kolmogorov-Smirnov (KS) test under this approach shows decreasing power as the structural shift level increases (see Figure 1 in Shao and Zhang (2010)).
- To enhance power in self-normalized structural break tests, Shao and Zhang (2010) introduced the G-test:
    - This method incorporates both backward and forward summation when constructing the self-normalizer.
    - Assume there is only one change-point, or adjustments to the formula are necessary based on the number of change-points. Computationally expensive!
- Zhang and Lavitas (2018) introduce a contrast-based self-normalized approach, which detects change-points through recursive scanning.

Introduction

- The specifications in Shao and Zhang (2010) and Zhang and Lavitas (2018) limit their applicability in sequential change-point detection.
- Chan, Ng, and Yau (2021) introduce the self-normalized sequential change-point detection scheme (SSMS), employing a self-normalized KS test for sequential monitoring based on Shao's (2010) SN.
- **Drawbacks:**
  - As the training set expands, mild or borderline structural breakpoints might be included due to Type II errors. Figure 1 in Shao and Zhang (2010) shows that the self-normalized KS test has a severe non-monotonic power issue, as the level of structural break increases, and the percentage of rejections decreases.
  - This would lead to underperformance in Chan, Ng, and Yau's (2021) scheme, as mild or borderline structural breakpoints are incorporated into the training sample.
- Here we introduce the adjusted-range based self-normalized sequential change-point monitoring scheme (RSMS) to rectify these issues.

Table of Contents

1. Introduction
2. Problem setup and sequential monitoring scheme
3. Asymptotic properties of the RSMS
4. Simulations
5. Empirical analysis
6. Conclusion
7. References

Quantity of interest

- Consider a d-dimensional sequence of stationary observations $\{X_t\}_{t=1}^n$.
- The mpirical distribution is $\widehat{\mathbf{F}}_{i,j}^d = \frac{1}{j-i+1}\sum_{t=i}^j \delta_{X_t}$, where δ_x assigns a mass of 1 at x. $\widehat{\mathbf{F}}_n^d$ represents the empirical distribution over the full sample period.
- $\theta_n = \mathcal{G}(\mathbf{F}^d)$ represents the quantity of interest.
- In practice, we use $\widehat{\theta}_n = \mathcal{G}(\widehat{\mathbf{F}}_n^d)$, which is a functional of the empirical distribution.
- $\widehat{\theta}_n$ includes linear statistics (marginal mean, variance, covariance), and estimators (LS, M-estimator, MM, MLE).

Quantity of interest

Assumption A.1

Assume $\{X_t\}_{t\in\mathbb{N}}$ *is strictly stationary with* $E\left\{\mathbf{IF}\left(X_t;\mathbf{F}^d\right)\right\}=0$, *and the weak convergence*

$$m^{-1/2}\sum_{t=1}^{\lfloor rm\rfloor}\mathbf{IF}\left(X_t;\mathbf{F}^d\right)\Longrightarrow\sqrt{\mathbf{\Sigma}\left(\mathbf{F}^d\right)}\mathbf{B}_q\left(r\right),$$

holds in $l^\infty([0,T+1],\mathbb{R}^q)$ *as* $m\to\infty$ *and* $T\in(0,\infty]$. *Here,* $\mathbf{B}_q(s)$ *is a* q*-dimensional standard Brownian motion, and* $\sqrt{\mathbf{\Sigma}\left(\mathbf{F}^d\right)}$ *is a* $q\times q$ *lower triangular matrix with nonnegative diagonal entries. The function space* $l^\infty([0,T+1],\mathbb{R}^q)$ *denotes the set of bounded* $\mathbb{R}^q$*-valued functions defined on* $[0,T+1]$ *endowed with the sup-norm. The long-run variance-covariance matrix is given by*

$$\mathbf{\Sigma}\left(\mathbf{F}^d\right)=\sum_{k=-\infty}^{\infty}\mathrm{Cov}\left\{\mathbf{IF}\left(X_0;\mathbf{F}^d\right),\mathbf{IF}\left(X_k;\mathbf{F}^d\right)\right\},$$

and is assumed to be positive definite.

Quantity of interest

Assumption A.2

The remainder term $\mathbf{Re}_{i,j}$ *arises from the equation:*

$$\mathcal{G}\left(\widehat{\mathbf{F}}_{i,j}^{d}\right)=\mathcal{G}\left(\mathbf{F}^{d}\right)+\frac{1}{j-i+1}\sum_{t=i}^{j}\mathbf{IF}\left(X_t,\mathbf{F}^{d}\right)+\mathbf{Re}_{i,j},$$

and it satisfies $\sup_{1\le i<j\le n}(j-i+1)\cdot|\mathbf{Re}_{i,j}|=o_p\left(n^{1/2}\right)$.

Problem setup

- **Change-Point Detection:** Observing the sequence $\{X_t\}$ to detect a change from θ_0 to $\theta_1 \neq \theta_0$ at unknown time t^*. For $t = 1$ to t^*, parameter is θ_0; for $t = t^*$ to n, parameter changes to θ_1.
- **Goal:** Timely detection of the change in parameter θ.
- **Parametric Model:** Sequence modeled by joint density f_θ across a class indexed by θ.
- **Estimation:** Parameter θ estimated by solving $\sum_{t=1}^{n} \psi(X_t, \theta) = 0$.

Problem setup

- **Training Sample:** First m observations.
- **Hypotheses:** $H_0 : \theta = \theta_0$ for all $t = 1, 2, \ldots, m + mT$ versus $H_1 : \theta$ changes at t^* to θ_1, $t^* = m + k^*$.
- **Monitoring Horizon:** mT where T is fixed; $mT \to \infty$ as $m \to \infty$.
- **Estimation Consistency:** $\widehat{\theta}_m \xrightarrow{p} \theta_0$ under suitable conditions.
- **Generalized Residuals vs. Generalized Forecast Errors:** No change-point implies $\{\psi(X_t, \widehat{\theta}_m)\}_{t=1}^{m}$ and $\{\psi(X_t, \widehat{\theta}_m)\}_{t=m+1}^{mT}$ are similar in a probabilistic sense.

Problem setup

- **CUSUM Statistic:** $S_m(k, \widehat{\theta}_m) = \sum_{t=m+1}^{m+k} \psi(X_t, \widehat{\theta}_m)$, expected to mimic Brownian motion when there is no change-points.
- **Temporal Dependence and Needs for LRV Estimation:** Generalized residuals and forecast errors often contain temporal dependencies, except in special cases like IID error terms, which necessitate LRV standardization, typically via a HAC estimator.
- **Challenges with HAC:** HAC robust tests can show poor size performance in finite samples with realistic dependencies. Alternative Methods such as the fixed-b asymptotics, block bootstrap, and subsampling, each requiring tuning parameters.

Problem setup

- **SSMS:** Introduced by Chan, Ng, and Yau (2021) using the SN approach from Shao (2010) for sequential change-point monitoring. SSMS utilizes a KS-type statistic for monitoring changes in the sequence.
- **Simulation Studies:** Shao and Zhang (2010) found that the self-normalized KS test statistic has decreasing power with increasing levels of structural shifts.
- **Stability and Sensitivity Issues:** Stability of the model based on $\{X_t\}_{t=1}^m$ is crucial. Mild breakpoints in the training sample can inflate the self-normalizer, reducing the sensitivity of the monitoring statistic.
- **Long-Term Implications:** With a fixed T and increasing m leading to $mT \to \infty$, “mild” structural changes will be inevitably included in the training sample due to Type II errors, jeopardizing the effectiveness of this approach in detecting change-points.

Problem setup

- We propose using Hong et al. (2024)'s adjusted-range-based self-normalization (SN). The adjusted-range statistic offers robust detection, even with "mild" structural changes in the training sample.
- $\psi\left(X_t, \widehat{\theta}_m\right)$ has temporal dependence.
  - We start from a consistent estimator for the variance-covariance matrix, e.g., the sample estimator.
  - Apply the square root-free Cholesky (LDL) decomposition to the sample variance-covariance matrix such that $\widehat{\mathbf{\Sigma}}_m = \widehat{\mathbf{C}}_m \widehat{\mathbf{D}}_m \widehat{\mathbf{C}}_m'$.
  - Define $\psi^\dagger$: $\psi^\dagger\left(X_t, \widehat{\theta}_m\right) = \widehat{\mathbf{C}}_m^{-1} \psi\left(X_t, \widehat{\theta}_m\right)$, which consists of no temporal dependence.

Problem setup

- Define

$$S_m^{\dagger}\left(k,\widehat{\theta}_m\right)=\left\{s_m^{\dagger(1)}\left(k,\widehat{\theta}_m\right),\ldots,s_m^{\dagger(d)}\left(k,\widehat{\theta}_m\right)\right\}'=\sum_{t=m+1}^{m+k}\psi^{\dagger}\left(X_t,\widehat{\theta}_m\right)\in\mathbb{R}^d, \tag{1}$$

for $k=1,2,\ldots,mT$.

- Set

$$\widetilde{S}_m^{\dagger}\left(k,\widehat{\theta}_m\right)=\left\{\widetilde{s}_m^{\dagger(1)}\left(k,\widehat{\theta}_m\right),\ldots,\widetilde{s}_m^{\dagger(d)}\left(k,\widehat{\theta}_m\right)\right\}'=\sum_{t=1}^{k}\psi^{\dagger}\left(X_t,\widehat{\theta}_m\right)\in\mathbb{R}^d, \tag{2}$$

for $k=1,2,\ldots,m$.

-

$$\mathbf{R}_m\left(\widehat{\theta}_m\right)=m^{-1/2}\mathrm{diag}\begin{pmatrix}\max_{1\le k\le m}\left(\widetilde{s}_m^{(1)}\left(k,\widehat{\theta}_m\right)\right)-\min_{1\le k\le m}\left(\widetilde{s}_m^{(1)}\left(k,\widehat{\theta}_m\right)\right)\\ \vdots\\ \max_{1\le k\le m}\left(\widetilde{s}_m^{(d)}\left(k,\widehat{\theta}_m\right)\right)-\min_{1\le k\le m}\left(\widetilde{s}_m^{(d)}\left(k,\widehat{\theta}_m\right)\right)\end{pmatrix}\in\mathbb{R}^d\times\mathbb{R}^d, \tag{3}$$

where $\mathrm{diag}\left(\cdot\right)$ creates a square matrix with the given vector as its diagonal entries.

Problem setup

- The adjusted-range based monitoring statistic is

$$\mathbb{M}_m^R(k) = \frac{S_m^\dagger\left(k, \widehat{\theta}_m\right)' \mathbf{R}_m\left(\widehat{\theta}_m\right)^{-2} S_m^\dagger\left(k, \widehat{\theta}_m\right)}{m\left(1+\frac{k}{m}\right)^2\left(\frac{k}{k+m}\right)^{2\gamma}}, \tag{4}$$

- The proposed monitoring scheme uses a stopping time criterion defined as:

$$\mathbb{T}_m^R = \begin{cases} \min\left\{k : \mathbb{M}_m^R(k) > c_R, 1 \le k \le mT\right\}, & \text{if such } k \text{ exists;} \\ mT+1, & \text{if } \mathbb{M}_m^R(k) \le c_R \text{ for all } 1 \le k \le mT, \end{cases} \tag{5}$$

where c_R is a decision boundary.

Problem setup

- The stopping rule consists of:
  1. **Initialization:** Choose T to set the monitoring duration and specify boundary functions to control type I error rate, ensuring it approaches the desired level α.
  2. **Early Termination:** Stop at the smallest k, where $\mathbb{M}_m^R(k) > c_R$ within $1 \leq k \leq mT$.
  3. **Completion without Termination:** If $\mathbb{M}_m^R(k) \leq c_R$ for all $1 \leq k \leq mT$, set the stopping time to $mT + 1$, indicating no change is detected.

The simulated asymptotic critical values for the RSMS for $T = 1, 2, 5, 10$ when $d = 2$.

	$\gamma = 0$								$\gamma = 0.15$							
	$T = 1$		$T = 2$		$T = 5$		$T = 10$		$T = 1$		$T = 2$		$T = 5$		$T = 10$	
$d \backslash \alpha$	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
1	2.1	1.5	2.7	2	3.4	2.5	3.9	2.8	2.7	2	3.3	2.5	3.9	2.9	4.3	3.2
2	3.2	2.4	5.2	4.1	10.9	8.3	21.1	15.9	4.8	3.7	8	6.2	19.2	14.7	44.6	33.1
3	4	3.3	6.5	5.1	13.8	10.8	26.6	21.1	5.8	4.7	9.6	7.9	23.8	18.4	53.8	41.7
4	4.8	3.9	7.6	6.2	16.2	12.8	30.7	24	6.9	5.7	11.5	9.2	28.1	22.2	63.4	49.7
5	5.5	4.5	8.9	7.3	18.3	14.8	35.3	28.6	7.8	6.5	12.7	10.4	29.9	24.5	71.3	55.3

Note: The number of Monte Carlo replications is set to 10,000. The Brownian motion simulations utilize $10{,}000 \times (m/T + 1)$ i.i.d. $N(0, 1)$ random variables. Here, $T = 1, 2$ represent short monitoring horizons, $T = 5$ corresponds to a medium horizon, and $T = 10$ represents a long monitoring horizon.

Table of Contents

1. Introduction
2. Problem setup and sequential monitoring scheme
3. Asymptotic properties of the RSMS
4. Simulations
5. Empirical analysis
6. Conclusion
7. References

Asymptotic properties under H_0

Assumption A.3

The true parameter θ_0 lies within the interior region of Θ.

Assumption A.4

$\mathbb{E}\left[\sup_{\theta\in\Theta}\|\psi\left(X_t,\theta\right)\|\right]<\infty$, *and* θ_0 *is the unique solution such that* $\mathbb{E}\left[\psi\left(X_t,\theta\right)\right]=0$. *Specifically, for any given constant* $\epsilon>0$, *there exists a constant* $\kappa>0$ *such that* $\|\theta-\theta_0\|>\epsilon$ *implies* $\mathbb{E}\left[\psi\left(X_t,\theta\right)\right]>\kappa$.

Asymptotic properties under H_0

Assumption A.5

$\mathbb{E}[\sup_{\theta\in\Theta} \|\psi(X_t,\theta_0)\|^{2+\delta}] < \infty$, *for some* $\delta > 0$, *and* $\{X_t\}$ *is a strong mixing sequence with mixing coefficients* α_k *satisfying* $\sum_{k=1}^{\infty} \alpha_k^{\delta/(2+\delta)} < \infty$.

Assumption A.6

$\psi(X_t,\theta)$ *is continuously differentiable with respect to* θ *in a neighborhood* V_{θ_0} *of* θ_0, *and* $\mathbb{E}\left(\sup_{\theta\in V_{\theta_0}} \|\partial\psi(X_t,\theta)/\partial\theta\|\right) < \infty$.

Asymptotic properties under H_0

Theorem 1

Under Assumptions A.1-A.6, we have $\widehat{\theta}_m = \theta_0 + o_p(m^{-1/2})$, *and furthermore, we have*

a) $\mathbf{R}_m^2\left(\widehat{\theta}_m\right) \xrightarrow{\mathcal{D}} \left(\boldsymbol{\Sigma}_\psi^{1/2}\right)\mathbf{R}\left(\boldsymbol{\Sigma}_\psi^{1/2}\right)'$ *as* $m \to \infty$, *where*

$$\mathbf{R} = \left[\operatorname{diag}\left\{\sup_{r\in(0,1]}\left(\mathbb{B}_d(r) - r\mathbb{B}_d(1)\right) - \inf_{r\in(0,1]}\left(\mathbb{B}_d(r) - r\mathbb{B}_d(1)\right)\right\}\right]^2,$$

$\mathbb{B}_d(r)$ *is a standard d-dimensional Brownian motion, and* $\mathbf{R}_m\left(\widehat{\theta}_m\right)$ *is defined in Equation (3), and* $\boldsymbol{\Sigma}_\psi = \mathbb{V}\mathrm{ar}\left(\psi\left(X_t, \theta_0\right)\right) = \sum_{k=-\infty}^{\infty}\mathbb{E}\left[\psi(X_t, \theta_0)\psi(X_{t+k}, \theta_0)'\right]$.

b) *The asymptotic size of the RSMS with decision boundary* c_R *for* $T < \infty$ *is*

$$\lim_{m\to\infty} P\left(\mathbb{T}_m^R \leq mT \,\middle|\, H_0\right) = P\left(\sup_{1\leq s\leq T}\frac{\mathbb{U}_d(s)'\mathbf{R}^{-1}\mathbb{U}_d(s)}{(1+s)^2\left(\frac{s}{s+1}\right)^{2\gamma}} > c_R\right), \tag{6}$$

where $\mathbb{U}_d(s) = \mathbb{B}_d(1+s) - (1+s)\mathbb{B}_d(1)$.

Asymptotic properties under H_0

- For $T = \infty$, if $\{X_t\}$ is a geometrically ρ-mixing sequence, meaning that the ρ-mixing coefficient satisfies

$$\rho(k) := \sup_{f \in \mathcal{L}^2(\mathcal{A}_0), g \in \mathcal{L}^2(\mathcal{B}_k)} |\mathrm{Corr}(f, g)| = O(a^k), \quad 0 < a < 1,$$

where $\mathcal{A}_0$ and $\mathcal{B}_k$ are the σ-fields generated by $\{X_t : t \leq 0\}$ and $\{X_t : t \geq k\}$, respectively, and $\mathcal{L}^2(\mathcal{F})$ is the space of square-integrable $\mathcal{F}$-measurable random variables, then

$$\begin{aligned} \lim_{m \to \infty} P\left(\mathbb{T}_m^R < \infty \,\middle|\, H_0\right) &= P\left(\sup_{1 \leq s < \infty} \frac{\mathbb{U}_d(s)' \mathbf{R}^{-1} \mathbb{U}_d(s)}{(1+s)^2 \left(\frac{s}{s+1}\right)^{2\gamma}} > c_R\right) \\ &= P\left(\sup_{0 < u \leq 1} \frac{\mathbb{B}_d^\dagger(u)' \mathbf{R}^{-1} \mathbb{B}_d^\dagger(u)}{u^{2\gamma}} > c_R\right), \end{aligned} \tag{7}$$

where $\mathbb{U}_d(s) = \mathbb{B}_d(1+s) - (1+s)\mathbb{B}_d(1)$, and $\mathbb{B}_d^\dagger(u)$ is a standard d-dimensional Brownian motion independent of $\mathbf{R}$.

Asymptotic properties under H_0

- The decision boundary c_R can be determined such that the asymptotic size is equal to a prespecified significance level α, i.e.

$$P\left(\sup_{1\le s\le T}\frac{\left[\mathbb{B}_d(1+s)-(1+s)\mathbb{B}_d(1)\right]'\mathbf{R}^{-1}\left[\mathbb{B}_d(1+s)-(1+s)\mathbb{B}_d(1)\right]}{(1+s)^2\left(\frac{s}{s+1}\right)^{2\gamma}}>c_R\right)=\alpha, \tag{8}$$

or

$$P\left(\sup_{0<u\le 1}\frac{\mathbb{B}_d^{\dagger}(u)'\mathbf{R}^{-1}\mathbb{B}_d^{\dagger}(u)}{u^{2\gamma}}>c_R\right)=\alpha. \tag{9}$$

Asymptotic properties under H_1

Assumption A.7

a) *Write the process as* $\{X_t^*\}_{t \geq t^*}$ *after the change-point, and assume that* $\mathbb{E}(\psi(X_t^*, \theta_0)) = c \neq 0$ *for some constant* $c \in \mathbb{R}^d$.

b) *Assume that* $\mathbb{E}(\sup_{\theta \in U_{\theta_0}} \|\psi(X_t^*, \theta)\|) < \infty$ *for some neighborhood* U_{θ_0} *of* θ_0.

Asymptotic properties under H_1

Assumption A.8

Under the alternative hypothesis $(H_1 : \theta_1 \neq \theta_0)$, *assume the change occurs at* $m + k^* = \lfloor mc \rfloor$ *for some* $c \in (1, T+1)$. *Moreover, assume the existence of two Brownian bridges,* $(\mathbb{B}_d(s) - s\mathbb{B}_d(1))$ *and* $(\mathbb{B}_d(s) - \mathbb{B}_d(c) - (s-c)\mathbb{B}_d(1))$, *such that the joint weak convergence*

$$\begin{pmatrix} \left\{ \frac{1}{\sqrt{m}} \sum_{t=1}^{\lfloor ms \rfloor} \mathbf{IF}\left(X_t, \theta_0\right) \right\}_{s\in(0,c]} \\ \left\{ \frac{1}{\sqrt{m}} \sum_{t=\lfloor mc \rfloor+1}^{\lfloor ms \rfloor} \mathbf{IF}\left(X_t^*, \theta_1\right) \right\}_{s\in(c,T+1]} \end{pmatrix} \overset{\mathcal{D}}{\Longrightarrow} \begin{pmatrix} \left\{ \sqrt{\boldsymbol{\Sigma}^{(1)}} \left(\mathbb{B}_d(s) - s\mathbb{B}_d(1)\right) \right\}_{s\in(0,c]} \\ \left\{ \sqrt{\boldsymbol{\Sigma}^{(2)}} \left(\mathbb{B}_d(s) - \mathbb{B}_d(c) - (s-c)\mathbb{B}_d(1)\right) \right\}_{s\in(c,T+1]} \end{pmatrix}$$

holds, where $\boldsymbol{\Sigma}^{(1)}$ *and* $\boldsymbol{\Sigma}^{(2)}$ *are positive definite matrices, defined as*

$$\boldsymbol{\Sigma}^{(\ell)} = M^{-1}\boldsymbol{\Sigma}_{\psi}^{(\ell)} \left(M^{-1}\right)' = M^{-1} \sum_{k=-\infty}^{\infty} E\left[\psi^{(\ell)}(X_t, \theta_0)\psi^{(\ell)}(X_{t+k}, \theta_0)'\right] \left(M^{-1}\right)', \quad \ell = 1, 2.$$

Asymptotic properties under H_1

Theorem 2

For the RSMS with a decision boundary c_R satisfying Equations (8) or (9) for a given significance level $\alpha \in (0,1)$, and under Assumptions A.7 and A.8, the asymptotic power of the RSMS is 1, i.e.,

$$\lim_{m\to\infty} P\left(T_m^R \leq mT \middle| H_1 \right) = 1 \text{ for } T < \infty,$$

and

$$\lim_{m\to\infty} P\left(T_m^R \leq mT \middle| H_1 \right) = 1 \text{ for } T = \infty.$$

Table of Contents

1 Introduction

2 Problem setup and sequential monitoring scheme

3 Asymptotic properties of the RSMS

4 Simulations

5 Empirical analysis

6 Conclusion

7 References

Online Change-Point Detection for Conditional Mean Using the AR Model

- We consider online change-point detection for the conditional mean using the AR model, as well as online change-point detection for both the conditional mean and variance using the PAR model.
- We use Monte Carlo simulations to assess model accuracy with metrics such as Type I error, rejection rate, and average run length (ARL).

Online Change-Point Detection for Conditional Mean Using the AR Model

Data generating processes

- **DGP1: Simple Homoskedastic Errors**. We consider a bivariate VAR(1) model given by $X_t = \Psi X_{t-1} + \varepsilon_t$, where $\Psi = \begin{pmatrix} 0.5 & 0.0 \\ 0.0 & 0.5 \end{pmatrix}$, and $\{\varepsilon_t\}$ follows an IID multivariate normal distribution $\mathrm{MN}\left(\mathbf{0}, I_2\right)$, where MN stands for a multivariate normal distribution and I_2 is a 2×2 identity matrix.
- **DGP2: VAR with Homoskedastic Errors**. Everything else remains the same as in DGP1, except that we now allow for some cross-dependence in $\{X_t\}$ and $\{\varepsilon_t\}$, namely $\Psi = \begin{pmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{pmatrix}$, and $\{\varepsilon_t\} \sim \mathrm{MN}\left(\mathbf{0}, \mathbf{\Sigma}_\varepsilon\right)$, with $\mathbf{\Sigma}_\varepsilon = \begin{pmatrix} 1.0 & 0.1 \\ 0.1 & 1.0 \end{pmatrix}$.

Online Change-Point Detection for Conditional Mean Using the AR Model

Data generating processes

- **DGP3: VAR with Conditional Heteroskedastic Errors**. The settings resemble DGP2, though we include conditional heteroskedasticity in the errors. Specifically, the sequence $\{\varepsilon_t\}$ follows a GARCH(1,1) process, such that

$$\varepsilon_t = \boldsymbol{\Sigma}_t^{1/2} e_t, \quad \sigma_{i,t}^2 = (1 - \alpha_1 - \beta_1) + \alpha_1^2 \varepsilon_{i,t-1}^2 + \beta_1 \sigma_{i,t-1}^2, \quad i = 1, 2$$

  where $\boldsymbol{\Sigma}_t = \begin{pmatrix} \sigma_{1,t}^2 & 0 \\ 0 & \sigma_{2,t}^2 \end{pmatrix}$, $(\alpha_1, \beta_1) = (0.1, 0.2)$, and $\{e_t\}$ is a vector of innovations following IID $\mathrm{MN}(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon)$.

- **DGP4: VAR with Unconditional Heteroskedastic Errors**. In this instance, we consider a jump in volatility:

$$\sigma_{i,t}^2 = \sigma_0^2 \left[1 + \delta I(t > m/2)\right], \quad \sigma_0 = 1, \quad \delta = 0.2, \quad i = 1, 2.$$

  Everything else remains identical to DGP 2.

Online Change-Point Detection for Conditional Mean Using the AR Model

Testing Constancy of Mean Level

- The purpose is to test the constancy of the mean level.
- Null hypothesis: $H_0 : \mu = \mu_0$
- Alternative hypothesis: H_1: H_0 is not true
- Set $\psi(X_t, \widehat{\mu}_m) = X_t - \widehat{\mu}_m$, where

$$\widehat{\mu}_m = \frac{1}{m} \sum_{t=1}^{m} X_t$$

- The function $\psi(X_t, \widehat{\mu}_m)$ is used to construct the RSMS, SSMS, and CSMS statistics.

Online Change-Point Detection for Conditional Mean Using the AR Model

Stopping Times and Monitoring Horizon

- The RSMS stopping time for the mean is defined as:

$$C_m = \left\{ \min \left\{ k : \frac{\left|\sum_{t=m+1}^{m+k} (X_t - \widehat{\mu}_m)\right|}{\mathbf{R}_m(\widehat{\mu}_m)} > m^{1/2} \left(1 + \frac{k}{m}\right) c_R \right\}, \, mT + 1 \right\}$$

- The stopping times for SSMS and CSMS are defined similarly, with $\mathbf{R}_m(\widehat{\mu}_m)$ replaced by the square root of either Shao's 2010 self-normalizer or the LRV estimator, respectively.
- For CSMS, the LRV is estimated using Silverman's Rule of Thumb for bandwidth selection.
- Monitoring horizons: mT, with $T = 1, 2, 5, 10$ under H_0, and $T = 1, 2, 5$ under H_1.
- Note: $T = 10$ approximates the open-end scheme ($T = \infty$), which is practically infeasible.

Online Change-Point Detection for Conditional Mean Using the AR Model

Simulated type I error rates for the RSMS (R), SSMS (S), and CSMS (C) for a change in the mean ($\gamma = 0$ and 0.15).

		T=1						T=2						T=5						T=10					
		R		S		C		R		S		C		R		S		C		R		S		C	
		5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
		$\gamma = 0$																							
DGP1	$m = 100$	9.0	14.4	2.6	5.1	12.4	18.8	10.5	16.2	3.0	5.8	9.0	13.1	8.2	12.6	3.6	5.9	2.2	3.0	7.4	14.0	2.6	6.4	0.0	0.3
	$m = 500$	4.3	7.3	1.5	2.5	5.0	9.0	3.3	6.8	1.1	2.0	1.8	3.7	4.2	8.6	1.4	2.9	0.1	0.3	3.4	5.3	1.4	2.6	0.0	0.0
	$m = 1000$	3.9	6.2	1.5	3.2	3.3	5.9	3.1	5.4	0.7	2.0	1.9	3.3	2.9	7.8	2.0	3.7	0.1	0.3	2.8	5.1	0.9	2.4	0.0	0.0
DGP2	$m = 100$	10.8	16.0	2.8	4.4	11.9	18.1	11.4	16.9	3.0	6.2	8.0	12.9	10.9	16.7	4.0	6.3	1.7	3.0	10.1	15.5	2.8	5.6	0.2	0.3
	$m = 500$	5.0	8.4	1.4	2.6	4.6	8.5	5.4	10.0	1.1	2.0	1.8	4.0	5.8	8.8	1.4	2.9	0.1	0.3	3.6	7.2	1.6	2.4	0.0	0.0
	$m = 1000$	4.4	7.0	1.5	3.2	3.0	5.4	4.7	8.0	0.7	2.0	2.0	3.4	4.2	7.1	2.0	3.8	0.1	0.3	3.8	6.8	0.9	2.3	0.0	0.0
DGP3	$m = 100$	10.9	16.3	2.5	4.9	11.8	17.8	11.2	17.4	2.9	6.3	7.9	12.7	10.5	16.1	3.9	5.8	1.7	3.0	10.7	15.2	2.9	5.8	0.1	0.3
	$m = 500$	4.8	8.6	1.2	2.8	5.5	8.4	5.3	10.0	1.0	1.9	1.8	4.0	5.8	9.3	1.4	2.7	0.1	0.4	4.3	7.1	1.7	2.2	0.0	0.0
	$m = 1000$	4.5	6.9	1.6	3.0	2.9	5.9	4.7	8.1	1.0	2.5	1.8	3.6	4.5	6.8	1.9	3.9	0.1	0.2	3.9	7.3	1.1	2.3	0.0	0.0
DGP4	$m = 100$	13.6	19.2	3.7	5.5	15.5	21.8	13.0	19.9	3.5	7.1	10.1	15.5	12.5	17.7	4.1	7.1	1.5	4.0	11.3	16.7	3.1	6.7	0.1	0.2
	$m = 500$	6.8	11.0	2.0	4.0	6.5	11.4	6.6	12.2	1.3	2.5	2.4	5.3	6.1	10.8	1.7	3.0	0.1	0.4	4.2	7.9	1.5	2.7	0.0	0.0
	$m = 1000$	5.4	8.9	1.8	3.7	4.4	7.6	5.2	9.6	1.2	2.8	2.6	4.2	4.4	8.1	2.2	4.6	0.2	0.4	4.3	7.6	1.3	2.9	0.0	0.0
		$\gamma = 0.15$																							
DGP1	$m = 100$	14.1	20.5	0.8	1.5	14.7	20.9	14.9	20.5	1.1	1.9	8.2	13.5	11.4	18.1	0.9	2.2	1.7	2.6	12.5	19.1	1.0	1.8	0.0	0.0
	$m = 500$	6.6	11.8	0.1	0.8	6.5	10.3	6.4	12.0	0.2	0.9	1.7	3.5	7.7	11.5	0.3	0.8	0.1	0.3	6.1	11.4	0.4	1.0	0.0	0.0
	$m = 1000$	6.4	9.8	0.5	0.8	4.0	6.6	5.1	8.6	0.3	0.5	1.7	3.1	6.4	10.6	0.7	1.4	0.1	0.2	5.8	10.0	0.0	0.5	0.0	0.0
DGP2	$m = 100$	15.7	21.7	1.0	2.1	13.7	20.0	15.6	22.2	1.3	1.9	8.3	13.0	15.3	20.5	1.0	2.5	1.4	2.3	14.6	21.4	1.1	2.2	0.0	0.3
	$m = 500$	8.5	14.0	0.1	0.7	6.6	9.9	9.1	14.8	0.2	0.9	1.2	3.3	9.4	15.2	0.4	0.9	0.1	0.3	7.6	12.6	0.5	1.1	0.0	0.0
	$m = 1000$	7.1	11.7	0.5	0.8	3.7	6.6	7.5	11.9	0.3	0.5	1.7	3.2	6.3	11.4	0.7	1.3	0.1	0.2	6.8	12.6	0.0	0.7	0.0	0.0
DGP3	$m = 100$	16.0	21.6	0.8	1.5	14.1	19.9	15.9	22.5	1.3	2.0	7.8	12.0	14.8	20.1	1.2	2.4	1.1	2.2	14.1	20.6	1.2	2.7	0.1	0.3
	$m = 500$	8.4	13.9	0.1	0.8	6.5	9.2	9.4	14.8	0.3	0.8	1.5	3.3	9.4	14.9	0.5	1.1	0.1	0.3	7.4	12.6	0.5	0.9	0.0	0.0
	$m = 1000$	7.0	11.5	0.5	0.8	3.7	6.8	7.7	12.6	0.4	0.7	1.6	3.2	6.8	11.6	0.6	1.0	0.1	0.2	6.9	11.8	0.1	0.5	0.0	0.0
DGP4	$m = 100$	18.8	26.6	1.3	2.6	18.2	24.9	18.5	25.8	1.5	2.6	10.1	16.0	16.4	22.4	1.5	2.6	1.3	2.7	15.9	22.4	1.1	2.4	0.1	0.1
	$m = 500$	11.3	17.6	0.3	1.1	8.5	13.6	11.8	18.5	0.2	1.0	2.2	5.7	11.7	18.1	0.3	0.8	0.2	0.3	8.0	14.2	0.6	1.4	0.0	0.0
	$m = 1000$	9.1	14.7	0.6	0.8	5.8	9.9	9.5	14.8	0.3	1.0	2.3	4.1	8.3	14.5	0.6	1.3	0.1	0.2	8.2	13.4	0.1	1.0	0.0	0.0

Online Change-Point Detection for Conditional Mean Using the AR Model

Type I Error Control: RSMS, SSMS, CSMS

- Type I error rates evaluated for RSMS, SSMS, and CSMS under the null hypothesis.
- **RSMS:** Type I errors cluster near nominal 5% and 10% levels, especially for moderate/large m (500, 1000); stable across all T.
- **SSMS:** Consistently conservative; type I error rates almost always below nominal, sometimes close to or below 1% for large m and T.
- **CSMS:** Most conservative; realized type I errors frequently at or near zero for moderate/large m and T.

Online Change-Point Detection for Conditional Mean Using the AR Model

Effect of γ and Overall Conclusions

- **Impact of γ:**
  - Increasing γ from 0 to 0.15 makes RSMS more liberal (size above nominal), SSMS even more conservative, and CSMS largely unaffected.
- **Conclusion:**
  - RSMS provides the most reliable size control, especially for $m = 500$ or 1000 and $T = 10$.
  - SSMS and CSMS remain overly conservative and may be too restrictive for practical use.

Online Change-Point Detection for Conditional Mean Using the AR Model

Empirical Rejection Rates under H_1

- Evaluate power of RSMS, SSMS, and CSMS under alternative hypothesis H_1
- Data-generating process modified to introduce a change-point at $t^* = m + k^*$
- Training sample size fixed at $m = 500$ for balance between size and efficiency
- Magnitude of structural change: $\Delta \in \{0.25, 0.50, 0.75, 1.00\}$
- Monitoring periods: $T = 1, 2, 5$ (focus on shorter horizons due to CSMS size issues)

Online Change-Point Detection for Conditional Mean Using the AR Model

Structural Break Scenarios

(i) Abrupt Change

$$Y_t = \begin{cases} X_t, & 1 \leq t \leq t^* \\ \Delta + X_t, & t^* < t \leq m + mT \end{cases}$$

(ii) Smooth Change

$$Y_t = \begin{cases} X_t, & 1 \leq t \leq t^* \\ \Delta \cdot \frac{t}{m+mT} + X_t, & t^* < t \leq m + mT \end{cases}$$

Online Change-Point Detection for Conditional Mean Using the AR Model

Key Considerations

- Both abrupt and smooth structural breaks are examined to assess robustness of monitoring schemes
- Results focus on early detection ability given restricted monitoring windows ($T = 1, 2, 5$)
- Performance metrics: empirical rejection rates (power) for each scheme across varying Δ

Online Change-Point Detection for Conditional Mean Using the AR Model

Rejection percentages of the RSMS (R), SSMS (S), and CSMS (C) for detecting a change in the mean with $m = 500$ and $k^* = 50, 200$ ($\gamma = 0$, Type i and ii).

		$k^*=50$												$k^*=200$											
		DGP1			DGP2			DGP3			DGP4			DGP1			DGP2			DGP3			DGP4		
T	Δ	R	S	C	R	S	C	R	S	C	R	S	C	R	S	C	R	S	C	R	S	C	R	S	C
Type i - abrupt change																									
$T=1$	0.25	45.9	40.0	40.5	35.1	21.9	12.4	12.8	11.8	59.4	37.0	36.9	33.8	20.1	18.9	19.0	18.3	9.2	6.7	6.3	6.3	29.1	18.5	18.6	19.0
$T=1$	0.5	97.6	91.9	92.1	84.4	73.0	49.7	50.0	42.9	99.4	92.4	92.5	85.6	71.9	60.2	60.0	50.1	37.4	22.9	23.1	19.2	83.0	57.2	56.7	48.2
$T=1$	0.75	100.0	99.9	99.9	99.0	93.6	82.4	82.6	74.4	100.0	100.0	99.9	99.5	97.2	91.6	91.8	83.8	72.0	48.8	48.6	41.6	99.2	91.8	91.6	84.4
$T=1$	1	100.0	100.0	100.0	100.0	98.2	94.9	95.0	90.7	100.0	100.0	100.0	100.0	99.9	99.1	99.2	97.0	90.5	73.6	73.4	63.0	100.0	99.6	99.6	97.9
$T=2$	0.25	54.1	46.0	46.1	39.6	32.5	18.7	18.6	16.1	66.7	41.5	41.9	35.5	37.0	32.6	32.6	28.6	21.1	12.5	12.3	10.9	47.6	28.7	29.2	24.9
$T=2$	0.5	99.4	96.9	96.9	92.4	86.6	66.5	66.8	58.4	100.0	97.7	97.8	92.4	95.7	89.3	89.4	81.2	74.0	51.5	51.6	43.2	98.7	87.4	87.7	79.0
$T=2$	0.75	100.0	100.0	100.0	99.8	98.4	90.8	90.4	86.2	100.0	100.0	100.0	100.0	100.0	99.5	99.4	98.8	95.2	82.3	82.1	75.0	100.0	99.9	99.9	99.1
$T=2$	1	100.0	100.0	100.0	100.0	99.9	98.0	98.0	95.8	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.0	94.2	94.2	91.2	100.0	100.0	100.0	100.0
$T=5$	0.25	36.4	32.6	32.4	25.4	43.9	26.5	26.9	22.3	49.1	25.0	25.0	19.4	30.2	26.2	26.5	20.7	38.5	22.3	23.1	18.6	40.6	19.8	19.9	16.2
$T=5$	0.5	98.6	94.3	94.2	87.7	91.0	77.0	77.1	69.4	100.0	95.3	95.2	87.6	97.3	90.6	90.6	82.9	87.7	72.3	72.2	63.6	99.9	91.2	90.8	80.3
$T=5$	0.75	100.0	100.0	100.0	99.7	99.4	95.5	95.8	92.1	100.0	100.0	100.0	100.0	100.0	99.9	99.9	99.4	99.1	93.7	93.6	89.5	100.0	100.0	100.0	99.8
$T=5$	1	100.0	100.0	100.0	100.0	99.9	99.6	99.6	98.7	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.9	99.1	99.1	97.7	100.0	100.0	100.0	100.0
Type ii - smooth change																									
$T=1$	0.25	26.6	23.8	24.6	22.0	12.1	8.0	8.0	7.7	36.8	23.2	23.3	22.7	14.7	15.0	14.9	14.4	6.7	5.4	4.9	5.4	21.4	14.6	14.9	16.2
$T=1$	0.5	84.2	71.5	71.9	62.1	49.1	31.8	31.8	26.7	91.4	71.2	70.9	61.3	52.6	44.5	44.8	37.6	26.3	14.6	14.8	12.6	66.2	40.9	40.8	36.5
$T=1$	0.75	99.7	96.4	96.4	91.8	83.0	60.1	59.8	51.2	100.0	97.2	97.1	92.6	90.3	78.5	79.2	69.4	56.7	36.8	37.1	31.1	94.9	78.6	78.6	68.3
$T=1$	1	100.0	100.0	99.9	99.0	93.9	83.7	83.5	75.0	100.0	100.0	99.9	99.6	99.5	95.4	95.4	90.1	80.8	56.8	57.5	48.1	100.0	96.4	96.2	90.9
$T=2$	0.25	22.3	21.1	21.6	19.0	11.7	7.6	6.9	7.1	29.2	19.0	19.0	16.9	17.2	17.6	17.8	15.8	9.5	5.8	5.4	5.8	23.5	15.2	15.2	14.0
$T=2$	0.5	83.4	71.4	71.2	61.3	54.6	33.3	33.3	27.4	91.1	67.5	67.1	56.5	73.3	61.6	61.5	51.3	44.7	26.5	26.5	21.8	83.7	56.1	56.1	46.4
$T=2$	0.75	99.4	97.0	97.0	92.0	86.7	66.4	66.5	57.7	100.0	97.7	97.6	91.7	97.7	93.3	93.4	86.5	80.4	57.5	57.4	49.4	99.6	93.3	92.9	84.7
$T=2$	1	100.0	99.9	99.8	99.2	96.5	86.3	85.9	79.0	100.0	100.0	100.0	99.6	100.0	99.4	99.3	98.1	94.1	79.9	79.6	71.9	100.0	99.7	99.6	98.8
$T=5$	0.25	6.6	6.9	7.0	5.7	13.1	8.4	8.3	8.0	9.4	4.5	4.5	3.6	6.0	6.5	6.6	5.5	12.6	8.2	8.2	7.9	9.1	4.2	4.4	3.4
$T=5$	0.5	53.0	44.1	43.7	33.6	54.9	33.2	33.6	26.8	66.4	35.4	35.6	26.0	50.0	42.0	41.7	32.1	53.1	31.4	32.3	26.2	63.4	33.0	33.2	24.4
$T=5$	0.75	95.2	85.4	85.6	75.1	84.0	66.9	66.5	56.8	99.4	84.4	84.0	70.5	94.3	83.8	83.8	72.6	82.6	64.6	64.7	54.9	99.2	81.7	81.5	68.0
$T=5$	1	99.9	99.0	99.0	95.7	96.3	86.2	85.3	79.0	100.0	99.4	99.4	96.5	99.9	98.7	98.6	95.2	96.0	85.0	84.4	77.8	100.0	99.1	99.0	95.8

Online Change-Point Detection for Conditional Mean Using the AR Model

Rejection percentages of the RSMS (R), SSMS (S), and CSMS (C) for detecting a change in the mean with $m = 500$ and $k^* = 50, 200$ ($\gamma = 0.15$, Type i and ii).

$k^* = 50$

T	Δ	DGP1 R	DGP1 S	DGP1 C	DGP2 R	DGP2 S	DGP2 C	DGP3 R	DGP3 S	DGP3 C	DGP4 R	DGP4 S	DGP4 C
Type i - abrupt change													
$T=1$	0.25	48.2	42.5	42.6	37.6	10.3	5.8	5.4	5.2	61.3	39.3	39.1	36.7
$T=1$	0.5	97.9	92.3	92.4	85.6	53.7	33.3	33.2	27.5	99.4	92.9	92.8	86.5
$T=1$	0.75	100.0	100.0	99.9	99.1	87.0	65.5	65.8	55.9	100.0	100.0	100.0	99.6
$T=1$	1	100.0	100.0	100.0	100.0	95.6	87.5	87.7	79.6	100.0	100.0	100.0	100.0
$T=2$	0.25	51.8	44.8	44.8	38.1	17.1	8.6	8.5	7.1	64.1	39.2	39.6	33.5
$T=2$	0.5	99.1	96.4	96.5	91.7	73.7	50.3	49.7	40.9	100.0	97.4	97.2	91.0
$T=2$	0.75	100.0	100.0	100.0	99.8	95.1	82.4	81.9	74.9	100.0	100.0	100.0	100.0
$T=2$	1	100.0	100.0	100.0	100.0	99.0	94.2	94.2	90.9	100.0	100.0	100.0	100.0
$T=5$	0.25	30.8	27.4	27.5	21.4	27.1	13.3	14.1	10.9	41.2	19.2	19.5	15.3
$T=5$	0.5	97.8	92.2	92.3	84.7	81.2	62.2	62.5	53.7	99.9	92.8	92.8	82.2
$T=5$	0.75	100.0	99.9	99.9	99.6	97.9	89.7	89.3	83.6	100.0	100.0	100.0	99.9
$T=5$	1	100.0	100.0	100.0	100.0	99.6	97.7	97.9	95.2	100.0	100.0	100.0	100.0
Type ii - smooth change													
$T=1$	0.25	28.7	26.0	26.3	24.2	5.2	3.0	2.6	3.0	38.7	25.1	25.1	25.1
$T=1$	0.5	85.1	72.9	73.1	64.3	30.0	15.3	15.7	11.8	91.7	72.3	72.3	62.9
$T=1$	0.75	99.7	96.5	96.7	92.2	65.6	42.0	42.0	34.5	100.0	97.2	97.3	93.2
$T=1$	1	100.0	100.0	99.9	99.1	87.4	67.0	67.0	56.5	100.0	100.0	99.9	99.6
$T=2$	0.25	19.9	19.6	20.1	17.9	5.8	3.0	2.5	2.8	27.1	17.1	16.7	15.3
$T=2$	0.5	81.7	69.2	68.8	58.6	36.6	18.5	18.8	14.7	89.8	63.9	64.3	53.6
$T=2$	0.75	99.2	96.4	96.5	91.3	73.8	50.3	49.9	40.4	100.0	97.0	96.9	90.3
$T=2$	1	100.0	99.8	99.8	99.1	91.5	74.8	74.4	66.2	100.0	100.0	100.0	99.4
$T=5$	0.25	4.8	5.1	5.3	4.3	6.1	3.5	3.6	3.1	6.2	2.8	3.1	2.6
$T=5$	0.5	44.7	37.2	37.5	27.4	37.3	18.8	18.9	14.3	58.2	28.1	27.8	20.2
$T=5$	0.75	93.1	81.7	81.7	69.7	72.0	49.1	49.5	39.9	98.8	78.0	77.8	63.7
$T=5$	1	99.8	98.2	98.1	94.3	89.6	74.8	74.4	66.0	100.0	99.0	98.8	94.9

$k^* = 200$

T	Δ	DGP1 R	DGP1 S	DGP1 C	DGP2 R	DGP2 S	DGP2 C	DGP3 R	DGP3 S	DGP3 C	DGP4 R	DGP4 S	DGP4 C
Type i - abrupt change													
$T=1$	0.25	22.4	20.5	20.3	20.4	3.5	2.0	2.0	2.2	30.8	20.2	20.5	21.1
$T=1$	0.5	74.0	61.9	61.7	52.5	21.4	10.1	9.8	7.9	84.3	58.5	58.9	49.8
$T=1$	0.75	97.7	92.3	92.4	85.1	52.9	32.3	32.1	26.2	99.3	92.3	92.1	85.5
$T=1$	1	100.0	99.3	99.3	97.2	78.9	53.9	54.0	45.1	100.0	99.6	99.6	98.0
$T=2$	0.25	34.2	31.1	31.2	27.1	10.2	4.9	4.5	4.5	44.1	26.4	26.4	23.2
$T=2$	0.5	95.0	88.2	88.4	79.6	56.8	34.1	33.9	26.6	98.4	85.6	85.6	76.6
$T=2$	0.75	100.0	99.4	99.4	98.8	88.6	69.6	69.4	60.6	100.0	99.8	99.8	99.1
$T=2$	1	100.0	100.0	100.0	100.0	97.3	88.0	87.9	82.1	100.0	100.0	100.0	100.0
$T=5$	0.25	24.1	21.4	21.7	16.8	22.2	11.4	11.5	9.2	32.3	15.2	15.3	11.3
$T=5$	0.5	96.1	87.4	87.4	77.7	76.5	57.1	56.4	46.9	99.8	86.7	86.5	73.3
$T=5$	0.75	100.0	99.7	99.7	99.1	96.4	86.8	85.7	79.8	100.0	100.0	100.0	99.7
$T=5$	1	100.0	100.0	100.0	100.0	99.6	96.2	96.4	93.3	100.0	100.0	100.0	100.0
Type ii - smooth change													
$T=1$	0.25	16.5	16.2	16.3	16.4	2.8	1.5	1.5	1.7	22.9	16.0	16.4	18.0
$T=1$	0.5	55.2	46.3	47.0	40.1	12.7	6.2	6.5	5.5	67.8	42.1	42.2	38.1
$T=1$	0.75	91.0	79.8	80.4	71.0	36.7	20.4	20.1	15.0	95.3	79.3	79.8	69.9
$T=1$	1	99.6	95.9	95.8	91.0	62.7	39.6	40.3	32.6	100.0	96.6	96.5	91.7
$T=2$	0.25	15.8	16.4	16.6	15.3	4.3	2.2	2.2	2.3	21.4	13.2	13.2	12.6
$T=2$	0.5	70.1	59.4	59.4	49.3	28.8	14.1	13.8	10.6	81.4	53.0	53.1	43.5
$T=2$	0.75	97.3	92.6	92.3	85.1	65.2	40.1	40.2	32.0	99.6	91.4	91.4	82.4
$T=2$	1	100.0	99.4	99.2	98.0	86.7	66.7	66.7	57.1	100.0	99.6	99.6	98.4
$T=5$	0.25	4.5	4.8	4.9	3.9	5.6	3.3	3.2	2.9	5.6	2.6	3.0	2.5
$T=5$	0.5	42.1	35.1	35.0	25.9	35.2	17.5	17.9	13.6	54.7	25.9	25.6	18.5
$T=5$	0.75	91.8	79.1	79.8	67.1	70.4	47.2	47.5	37.9	97.8	75.0	74.7	60.1
$T=5$	1	99.6	97.8	97.7	93.1	88.6	73.2	72.6	64.0	100.0	98.3	98.5	93.2

Online Change-Point Detection for Conditional Mean Using the AR Model

Power Analysis: General Patterns

- Power increases with larger change magnitude (Δ) and is higher for abrupt vs. smooth changes.
- Early changes ($k^* = 50$) are easier to detect than later ones ($k^* = 200$).
- Increasing monitoring horizon T initially raises power, but excessive T reduces power due to noise accumulation.

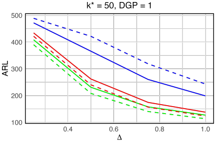
Online Change-Point Detection for Conditional Mean Using the AR Model

Power Analysis: Monitoring Schemes

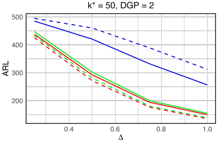
- **RSMS:** Consistently achieves highest or near-highest power across most scenarios, especially for large Δ.
- **SSMS:** Slightly lower power than RSMS, especially for small breaks or challenging DGPs.
- **CSMS:** Lowest power in most DGPs, but performs better under DGP 3.
- Introducing $\gamma = 0.15$ generally increases power, especially for early/abrupt changes and small Δ; largest gains for RSMS.
- For long horizons or smooth changes, power improvements from γ are limited or negative for SSMS/CSMS.

Online Change-Point Detection for Conditional Mean Using the AR Model

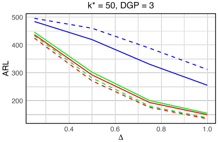
ARL Curves for Online Monitoring Schemes



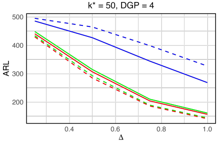
DGP1, $k^* = 50$



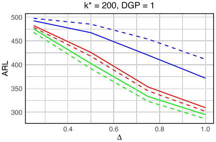
DGP2, $k^* = 50$



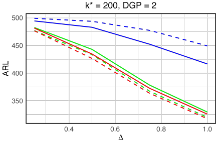
DGP3, $k^* = 50$



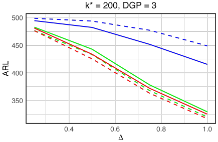
DGP4, $k^* = 50$



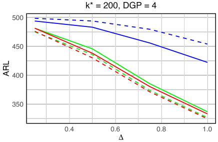
DGP1, $k^* = 200$



DGP2, $k^* = 200$



DGP3, $k^* = 200$



DGP4, $k^* = 200$

ARL curves of RSMS, SSMS, and CSMS for a change in the mean ($T = 1$, type I alternative). ARL curves of RSMS, SSMS, and CSMS for a change in the mean ($T = 1$, type I alternative). Solid lines: $\gamma = 0$, dashed: $\gamma = 0.15$.

Online Change-Point Detection for Conditional Mean Using the AR Model

Average Run Length (ARL) Insights

- **ARL** (Average Run Length) measures the expected delay before detecting a change—crucial in real-time risk, finance, and environmental monitoring.
- Simulations focus on $T = 1$ and abrupt changes (Type I); larger changes (Δ) yield shorter ARLs.
- **CSMS** achieves the shortest ARLs under DGP1, while **RSMS** is fastest for DGPs 2–4.
- Setting $\gamma = 0.15$ reduces ARLs for CSMS and RSMS (faster detection), but increases ARL for **SSMS**.
- **SSMS** consistently has the longest ARLs (slowest detection) across all DGPs and settings.
- **Conclusion:** **RSMS** delivers the best balance of timely detection and accurate size control, making it well suited for high-stakes, dynamic environments.

Online Change-Point Detection of Both Conditional Mean and Variance Using the PAR Model

The PAR(1) Model: Definition

- The Poisson Autoregressive process of order 1 (PAR(1)) is a count time series: $X_t \in \mathbb{Z}_+$.
- Recursion: $X_t = \alpha \circ X_{t-1} + \varepsilon_t$, where $\varepsilon_t \sim \mathrm{Pois}(\lambda)$.
- Binomial thinning: $\alpha \circ X_{t-1} = \sum_{i=1}^{X_{t-1}} Z_i$, with $Z_i \sim \mathrm{Bernoulli}(\alpha)$ i.i.d.
- Parameters: $\theta = (\alpha, \lambda)$, where $\alpha \in [0, 1)$ controls dependence and $\lambda > 0$ is the innovation rate.

Online Change-Point Detection of Both Conditional Mean and Variance Using the PAR Model

Motivation and Properties

- Marginal process is Poisson: mean = variance; enables joint monitoring of both parameters.
- Forecast error-based detection not directly applicable; use ML estimation for score-based monitoring.
- PAR(1) is relevant for practical applications (e.g., epidemic counts) and preserves count data properties (non-negativity, discreteness).
- Increased interest in count time series models since COVID-19, supporting practical value of PAR(1).

Online Change-Point Detection of Both Conditional Mean and Variance Using the PAR Model

Likelihood and Monitoring Procedure

- Conditional log-likelihood (given X_1):

$$l(\alpha, \lambda) = \sum_{t=2}^{n} \log P(X_t | X_{t-1})$$

- Transition probability:

$$P(X_t | X_{t-1}) = \sum_{i=0}^{X_t \wedge X_{t-1}} \binom{X_{t-1}}{i} \alpha^i (1-\alpha)^{X_{t-1}-i} \frac{e^{-\lambda} \lambda^{X_t - i}}{(X_t - i)!}$$

- Score functions: $\psi(X_t, \theta) = \left(\frac{\partial}{\partial \alpha} l, \frac{\partial}{\partial \lambda} l \right)'$
- Hypotheses: H_0: θ constant; H_1: change in θ.
- Use ML estimator $\widehat{\theta}_m$ from training sample; evaluate scores for monitoring (RSMS, SSMS, CSMS).

Online Change-Point Detection of Both Conditional Mean and Variance Using the PAR Model

Simulated type I error rates for RSMS (R), SSMS (S), and CSMS (C) under the PAR(1) null hypothesis.

	T=1						T=2						T=5						T=10					
	R		S		C		R		S		C		R		S		C		R		S		C	
	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
$\gamma = 0$																								
$m = 100$	9.2	12.3	3.9	7.1	11.5	15.4	8.7	13.9	3.1	5.8	8.1	11.6	8.3	12.9	3.4	6.3	2.1	3.9	8.5	12.1	3.3	5.5	0.8	1.3
$m = 500$	4.6	8.2	1.1	3.5	2.9	6.3	4.3	8.5	1.4	3.3	1.6	3.8	4.8	7.1	1.8	3.0	0.1	0.3	4.3	6.4	1.7	3.5	0.0	0.0
$m = 1000$	2.8	5.5	1.1	3.0	1.4	3.9	2.7	6.5	1.2	2.5	0.7	1.7	2.9	5.3	1.7	3.0	0.0	0.2	2.3	4.7	1.6	2.8	0.0	0.0
$\gamma = 0.15$																								
$m = 100$	13.4	18.1	1.5	2.7	13.3	17.5	13.4	19.5	1.1	2.4	8.6	12.0	12.1	18.1	1.5	2.8	2.3	3.1	12.8	18.1	1.5	2.8	0.6	1.1
$m = 500$	8.0	13.4	0.3	0.6	4.2	7.8	7.9	11.8	0.3	1.3	1.6	3.5	6.9	11.9	0.3	1.2	0.1	0.1	7.2	12.5	0.2	1.1	0.0	0.0
$m = 1000$	5.6	10.1	0.0	0.6	2.3	5.2	6.3	10.5	0.2	0.7	0.5	1.4	6.0	9.8	0.5	1.0	0.0	0.0	5.8	10.3	0.6	1.2	0.0	0.0

Online Change-Point Detection of Both Conditional Mean and Variance Using the PAR Model

Type I Error Rates: PAR(1) Model

- For $\gamma = 0$ and $m = 100$, CSMS and RSMS are closest to nominal size for small T; SSMS is most conservative at $T = 1, 2$.
- As T increases, CSMS becomes most conservative, while RSMS remains well-sized even for $T = 10$.
- Increasing m makes all methods more conservative; Type I error rates drop below nominal sizes.
- For $\gamma = 0.15$ and $m = 100$, CSMS and RSMS show inflated rejection rates at small T, but quickly become conservative as m grows; all methods under-reject for large m and T.
- Overall: CSMS and RSMS are well-sized or liberal for small samples; all methods (especially SSMS) become conservative as m or T increase.
- Practical focus: Report power for $T = 1, 2, 5$ and restrict ARL analysis to $T = 1$ for relevance to fast detection.

Online Change-Point Detection of Both Conditional Mean and Variance Using the PAR Model

Power Analysis: Change-Point Detection in PAR(1) Model

- Empirical rejection frequencies of monitoring schemes are evaluated under H_1.
- Simulations use $m = 500$, with a change-point at $t^* = m + k^*$, $k^* \in \{50, 200\}$.
- Two types of structural changes are considered:
  - **(i) Abrupt shift:** α_t and λ_t switch instantly from (α_1, λ_1) to (α_2, λ_2) after t^*.
  - **(ii) Smooth change:** α_t and λ_t evolve gradually after t^* via an exponential decay function.

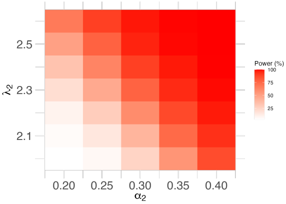
Online Change-Point Detection of Both Conditional Mean and Variance Using the PAR Model

Simulation Design: Parameter Grid and Scenarios

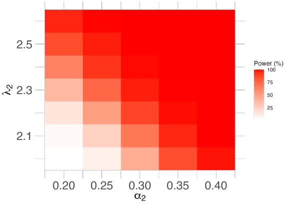
- Before change-point: $\alpha_1 = 0.2$, $\lambda_1 = 2$.
- After change-point: $\alpha_2 \in \{0.2, 0.25, \ldots, 0.35\}$, $\lambda_2 \in \{2.0, 2.1, \ldots, 2.5\}$.
- For smooth change, the transition is governed by rate $\delta = 0.05$.
- This two-dimensional grid systematically examines power under a wide range of signal strengths.
- Results reveal how detection power varies by change magnitude, abruptness, and delay (k^*).

Online Change-Point Detection of Both Conditional Mean and Variance Using the PAR Model

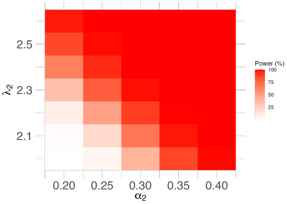
RSMS Heatmaps: Empirical Power Across T and γ



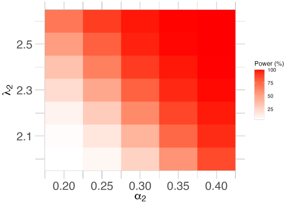
$\gamma = 0, \ T = 1$



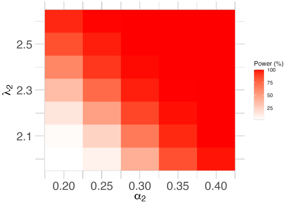
$\gamma = 0, \ T = 2$



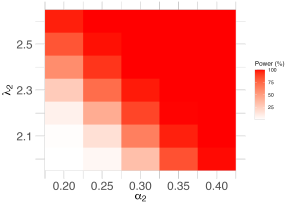
$\gamma = 0, \ T = 5$



$\gamma = 0.15, \ T = 1$



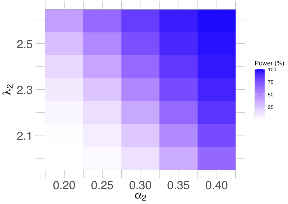
$\gamma = 0.15, \ T = 2$



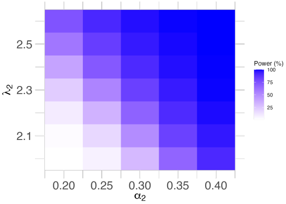
$\gamma = 0.15, \ T = 5$

Online Change-Point Detection of Both Conditional Mean and Variance Using the PAR Model

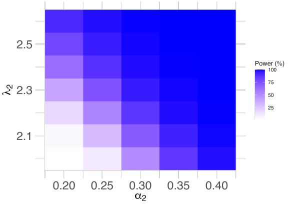
SSMS Heatmaps: Empirical Power Across T and γ



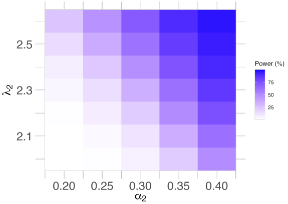
$\gamma = 0,\ T = 1$



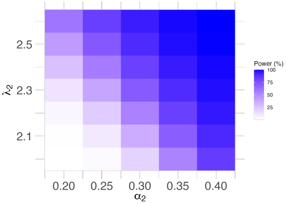
$\gamma = 0,\ T = 2$



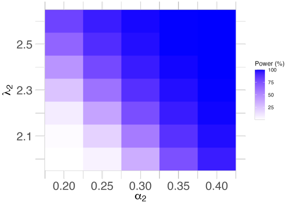
$\gamma = 0,\ T = 5$



$\gamma = 0.15,\ T = 1$



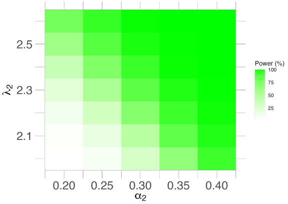
$\gamma = 0.15,\ T = 2$



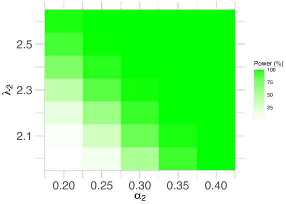
$\gamma = 0.15,\ T = 5$

Online Change-Point Detection of Both Conditional Mean and Variance Using the PAR Model

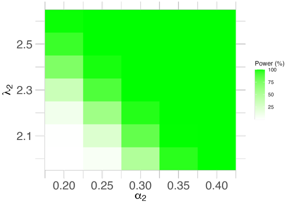
CSMS Heatmaps: Empirical Power Across T and γ



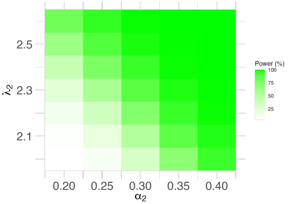
$\gamma = 0,\ T = 1$



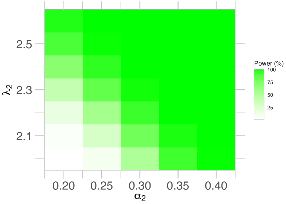
$\gamma = 0,\ T = 2$



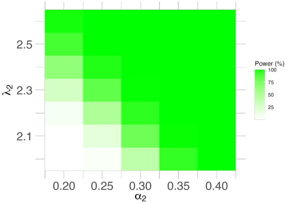
$\gamma = 0,\ T = 5$



$\gamma = 0.15,\ T = 1$



$\gamma = 0.15,\ T = 2$



$\gamma = 0.15,\ T = 5$

Online Change-Point Detection of Both Conditional Mean and Variance Using the PAR Model

Empirical Power Heatmaps: RSMS, SSMS, CSMS

- Heatmaps show empirical rejection rates for RSMS, SSMS, and CSMS across post-change parameter grids.
- Top row: $\gamma = 0$; Bottom row: $\gamma = 0.15$. Columns: $T = 1, 2, 3$.
- Color intensity reflects detection power (darker = higher rejection rate).
- RSMS and CSMS: Rapid, reliable power increase as the signal strengthens; upper-right corners approach 100% rejection.
- SSMS: Weaker performance; lighter heatmap regions indicate lower sensitivity and fewer cases of full rejection.

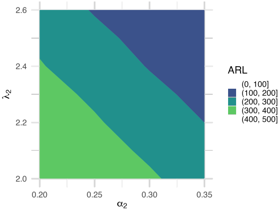
Online Change-Point Detection of Both Conditional Mean and Variance Using the PAR Model

ARL Contours: Early Detection Performance

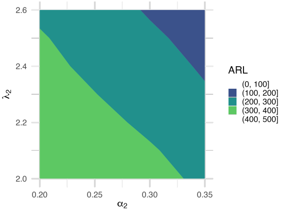
- Contour plots display ARL (Average Run Length) for RSMS, SSMS, and CSMS under Type (i) abrupt breaks.
- Scenario: $k^* = 50$, monitoring horizon $T = 1$ (early detection focus).
- ARL grid is divided into 5 intervals; darker colors represent faster detection (lower ARL).
- Results highlight the sensitivity of each method to signal strength, with rapid detection corresponding to higher color intensity.
- Observed patterns persist under smooth changes (Type (ii)), but with generally longer ARLs.

Online Change-Point Detection of Both Conditional Mean and Variance Using the PAR Model

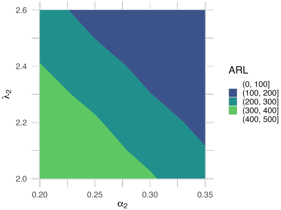
ARL Contours: RSMS, SSMS, CSMS ($k^* = 50$, $T = 1$)



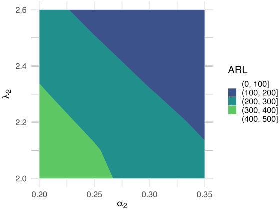
RSMS, $\gamma = 0$

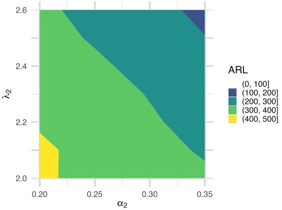


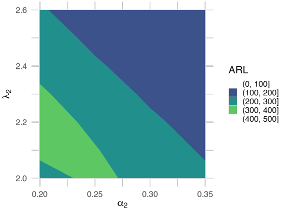
SSMS, $\gamma = 0$



CSMS, $\gamma = 0$







Online Change-Point Detection of Both Conditional Mean and Variance Using the PAR Model

ARL Contour Results: Sensitivity to Post-Change Parameters

- Each contour plot shows ARL across a grid of post-change parameters: α_2 (horizontal axis) and λ_2 (vertical axis).
- Across all schemes and both γ values, increasing α_2 and λ_2 results in lower ARL—stronger changes are detected faster.
- **Effect of γ:**
    - For RSMS and CSMS, setting $\gamma = 0.15$ further reduces ARL; dark (fast detection) regions expand.
    - For SSMS, $\gamma = 0.15$ leads to a yellow region (ARL 400–500), indicating slower detection.
- **Conclusion:** Early-stage detection is enhanced by $\gamma > 0$ for RSMS and CSMS, but not for SSMS.

Table of Contents

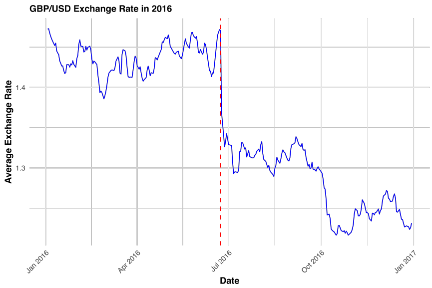
1. Introduction
2. Problem setup and sequential monitoring scheme
3. Asymptotic properties of the RSMS
4. Simulations
5. Empirical analysis
6. Conclusion
7. References

Empirical analysis

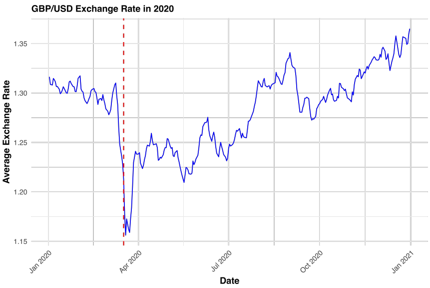
- We analyze USD/GBP exchange rates during 2016 (EU referendum), 2020, and 2022, using 5-minute intraday data from REFINITIV EIKON.
- We rescale exchange rates by 100 for numerical stability and standardize timestamps to $[0, 1]$.
- We smooth each curve using a cubic B-spline basis (number of basis functions $=$ min$\{21,\ 80\%\times$ [min obs/day]$\}$).
- The final sample comprises 312 valid daily curves for both 2019 and 2020.
- We apply functional principal component analysis (FPCA) to extract dominant variation modes for robust online detection.
- We compute functional principal component scores for the test set by projecting each daily curve onto eigenfunctions estimated from the training sample, ensuring comparability.
- This framework naturally extends online monitoring to functional time series (see, e.g., Sun et al. (2025) for FPCA-based KS test).

Empirical analysis

USD/GBP Exchange Rates During Major Events



2016 Brexit referendum. Dashed line: 23 June 2016.



2020 COVID-19 panic. Dashed line: 17 March 2020.

Daily average USD/GBP exchange rates during 2016 and 2020.

Empirical analysis

Summary of RSMS, SSMS, and CSMS test results (USD/GBP, 2016)

m	T	γ	Scheme	Statistic	Reject H_0?	Date
50	5	0.00	RSMS	213.92	Yes	12-07-2016
50	5	0.00	SSMS	459.22	Yes	16-08-2016
50	5	0.00	CSMS	186.74	Yes	15-07-2016
50	5	0.15	RSMS	225.94	Yes	26-07-2016
50	5	0.15	SSMS	459.22	Yes	20-09-2016
50	5	0.15	CSMS	197.23	Yes	01-08-2016
75	2	0.00	RSMS	41.83	Yes	18-07-2016
75	2	0.00	SSMS	132.23	Yes	10-08-2016
75	2	0.00	CSMS	57.21	Yes	17-07-2016
75	2	0.15	RSMS	47.24	Yes	22-07-2016
75	2	0.15	SSMS	132.23	Yes	01-09-2016
75	2	0.15	CSMS	64.61	Yes	21-07-2016

Empirical analysis

Summary of RSMS, SSMS, and CSMS test results (USD/GBP, 2016)

m	T	γ	Scheme	Statistic	Reject H_0?	Date
100	1	0.00	RSMS	22.28	Yes	17-07-2016
100	1	0.00	SSMS	133.54	Yes	27-07-2016
100	1	0.00	CSMS	39.88	Yes	14-07-2016
100	1	0.15	RSMS	27.43	Yes	18-07-2016
100	1	0.15	SSMS	133.54	Yes	08-08-2016
100	1	0.15	CSMS	49.10	Yes	15-07-2016
100	2	0.00	RSMS	188.70	Yes	19-07-2016
100	2	0.00	SSMS	1131.14	Yes	31-07-2016
100	2	0.00	CSMS	337.83	Yes	15-07-2016
100	2	0.15	RSMS	213.11	Yes	22-07-2016
100	2	0.15	SSMS	1131.14	Yes	14-08-2016
100	2	0.15	CSMS	381.53	Yes	19-07-2016

Empirical analysis

2016 USD/GBP Exchange Rate: Structural Break Detection Results

- Choices of m and T ensure post-Brexit period is within the monitoring window (sample size: 312 trading days).
- **RSMS consistently delivers the earliest detection**, especially for larger T.
- When H_0 is rejected, RSMS detects change-points earlier than SSMS, and often earlier than CSMS.
- Example: With $(m = 50, T = 5, \gamma = 0.00)$, RSMS signals on 12-07-2016; CSMS, 15-07-2016; SSMS, 16-08-2016.
- Example: With $(m = 75, T = 2, \gamma = 0.15)$, RSMS (22-07-2016) and CSMS (21-07-2016) outperform SSMS (01-09-2016).

Empirical analysis

Summary of RSMS, SSMS, and CSMS test results based on the USD/GBP exchange rate in 2019.

m	T	γ	Monitoring Scheme	Statistic	Reject H_0?	Date of Rejection
50	1	0.00	RSMS	2.38	Yes	04-04-2019
50	1	0.00	CSMS	3.09	Yes	02-04-2019
50	1	0.15	CSMS	3.92	Yes	15-04-2019
50	5	0.00	RSMS	21.48	Yes	05-08-2019
50	5	0.00	SSMS	56.59	Yes	10-10-2019
50	5	0.00	CSMS	27.88	Yes	04-08-2019
50	5	0.15	RSMS	23.00	Yes	27-08-2019
50	5	0.15	CSMS	29.86	Yes	23-08-2019

Empirical analysis

2019 USD/GBP Exchange Rate: Detection of Structural Breaks

- For $(m = 50, T = 5, \gamma = 0.00)$, RSMS signals a break on 5 August, nearly matching CSMS (4 August); SSMS alarms much later (10 October).
- Increasing γ to 0.15 does not improve early detection and often delays or prevents rejection (e.g., RSMS does not trigger for $T = 1$ with $\gamma = 0.15$).
- SSMS fails to detect any changes with $m = 50$, $T = 1$ for both γ values, and also fails for $m = 5, \gamma = 0.15$.
- These results highlight the limitations of SSMS when the training sample is short or contaminated by early pandemic shifts, which can inflate the self-normalizer and hinder detection.

Empirical analysis

Summary: Early Detection and Tuning Parameter Effects

- RSMS consistently provides the earliest detection and shortest ARL across all settings, aligning with simulation results.
- Evidence demonstrates the robustness and responsiveness of RSMS in dynamic environments (e.g., post-Brexit, COVID-19), making it well suited for practitioners needing timely anomaly detection.
- Setting $\gamma = 0.15$ does not generally improve detection sensitivity; its benefit is seen only under CSMS (empirical) and for both RSMS and CSMS (simulations), while it has negligible or slightly negative impact on SSMS.
- Overall, the effect of γ is highly scheme- and context-dependent, and should be considered with care.

Table of Contents

1. Introduction
2. Problem setup and sequential monitoring scheme
3. Asymptotic properties of the RSMS
4. Simulations
5. Empirical analysis
6. Conclusion
7. References

Conclusion

- We propose the RSMS: an adjusted-range-based self-normalized online change-point detection scheme that builds on the CUSUM framework and leverages robust self-normalization.
- RSMS requires no LRV estimation or tuning of bandwidth, kernel, or block size, and achieves accurate type I error control and consistent detection.
- Simulations show RSMS is robust across DGPs and break types, outperforming SSMS and CSMS in power and ARL.
- Empirical analysis on USD/GBP exchange rates (2016 Brexit, COVID-19) confirms RSMS's effectiveness and rapid detection of structural breaks.
- **Future work:** Apply RSMS to areas like climate data and social media; integrate with deep learning for improved adaptability in real-time detection.

Table of Contents

1. Introduction
2. Problem setup and sequential monitoring scheme
3. Asymptotic properties of the RSMS
4. Simulations
5. Empirical analysis
6. Conclusion
7. References

References I

- Berkes, István et al. (2004). "Sequential change–point detection in GARCH (p, q) models". In: *Econometric Theory* 20.6, pp. 1140–1167.
- Chan, Ngai Hang, Wai Leong Ng, and Chun Yip Yau (2021). "A self–normalized approach to sequential change–point detection for time series". In: *Statistica Sinica* 31.1, pp. 491–517.
- Chu, Chia-Shang James, Maxwell Stinchcombe, and Halbert White (1996). "Monitoring structural change". In: *Econometrica* 64.5, pp. 1045–1065.
- Gombay, Edit and Lajos Horváth (2009). "Sequential tests and change detection in the covariance structure of weakly stationary time series". In: *Communications in Statistics – Theory and Methods* 38.16–17, pp. 2872–2883.
- Gombay, Edit and Daniel Serban (2009). "Monitoring parameter change in AR(p) time series models". In: *Journal of Multivariate Analysis* 100.4, pp. 715–725.

References II

- Gut, Allan and Josef Steinebach (2002). "Truncated sequential change–point detection based on renewal counting processes". In: *Scandinavian Journal of Statistics* 29.4, pp. 693–719.
- Hong, Yongmiao et al. (2024). "Kolmogorov–Smirnov type testing for structural breaks: A new adjusted–range based self–normalization approach". In: *Journal of Econometrics* 238.2, p. 105603.
- Lobato, Ignacio N (2001). "Testing that a dependent process is uncorrelated". In: *Journal of the American Statistical Association* 96.455, pp. 1066–1076.
- Robbins, Herbert (1970). "Statistical methods related to the law of the iterated logarithm". In: *The Annals of Mathematical Statistics* 41.5, pp. 1397–1409.
- Shao, Xiaofeng (2010). "A self–normalized approach to confidence interval construction in time series". In: *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 72.3, pp. 343–366.

References III

- Shao, Xiaofeng and Xianyang Zhang (2010). "Testing for change points in time series". In: *Journal of the American Statistical Association* 105.491, pp. 1228–1240.
- Sun, Jiajing et al. (2025). "Structural stability of functional data-a new adjusted-range-based self-normalization approach". In: *Economics Letters*, p. 112350.
- Zeileis, Achim et al. (2005). "Monitoring structural change in dynamic econometric models". In: *Journal of Applied Econometrics* 20.1, pp. 99–121.
- Zhang, Ting and Liliya Lavitas (2018). "Unsupervised self–normalized change–point testing for time series". In: *Journal of the American Statistical Association* 113.522, pp. 637–648.

Feedback

Thank you! Any question or feedback.